# Polyhedral

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We work with a two mutually dual finite-dimensional vector spaces $N$ and $M$ over a linearly ordered field $\Bbbk$. This means, we have a bilinear pairing $\langle\cdot,\cdot\rangle\colon N\times M\to\Bbbk$ such that the induced maps $M\to\mathrm{Hom}(N,\Bbbk)$ and $N\to\mathrm{Hom}(N,\Bbbk)$ are both bijective (this is also known as a *perfect pairing*). Since $\Bbbk$ is linearly ordered, we can define the non-negative scalars $\Bbbk_{\geq 0}=\{t\in\Bbbk\mid t\geq 0\}\subseteq\Bbbk$. This is a *semifield*, i.e. a commutative semiring, in which every nonzero element has a multiplicative inverse.

**Definition 1.** A **hyperplane** is the perp space of a point.

A **halfspace** is the dual of a point. The dimension of the module the cone is a submodule of.

The **dimension** of a pointed cone.

A polyhedral cone is **full-dimensional** if its dimension is the same as the dimension of the ambient space.

The set of all $k$-dimensional faces of a polyheadral cone.

For each $k$ gives the number of $k$-dimensional faces of a polyheadral cone.

## 0.1 Polyhedral cones

**Definition 2.** A **pointed cone** $\sigma\subseteq N$ is a $\Bbbk_{\geq 0}$-submodule of $N$.

**Definition 3.** A **pointy cone** $\sigma\subseteq N$ is a $\Bbbk_{\geq 0}$-submodule of $N$ that does not contain a nontrivial linear space.

**Definition 4.** For a set $S\subseteq N$, the **cone generated by** $S$ is the pointed cone

$$\mathrm{Cone}(S):=\mathrm{Span}_{\Bbbk_{\geq 0}}(S)\subseteq N.$$

**Definition 5.** A pointed cone $\sigma\subseteq N$ is called **finitely generated**, if $\sigma=\mathrm{Cone}(S)$ for some finite set $S\subseteq N$.

**Definition 6.** For a set $S\subseteq M$, the **dual cone of** $S$ is the pointed cone

$$S^\vee:=\{x\in N\mid\forall y\in S,\langle x,y\rangle\geq 0\}\subseteq N.$$

**Definition 7.** Another word for finitely generated cone with a finitely generated canonical dual.

### 0.1.1 Examples

**Proposition 8.** *The zero-space $\{0\}\subseteq N$ is a polyhedral cone.*

*Proof.* If $S\subseteq M$ is a finite $\Bbbk$-basis of $M$, then $\{0\}=(S\cup -S)^\vee$. □

**Proposition 9.** *The full space is a polyhedral cone.*

**Proposition 10.** *Any halfspace is a polyhedral cone.*

**Proposition 11.** *Any linear subspace is a polyhedral cone.*

**Proposition 12.** *If $\sigma\subset N$ is finitely generated, then $\sigma^\vee\subset M$ is polyhedral.*

*Proof.* Let $\sigma=\mathrm{Cone}(S)$ for $S\subseteq N$. We need to show $\mathrm{Cone}(S)^\vee=S^\vee$. The inclusion "$\subseteq$" is obvious, as $S\subseteq\mathrm{Cone}(S)$. The opposite inclusion can be shown by induction on the general element of $\mathrm{Cone}(S)$. □

**Proposition 13** (Double dual)**.** *If $\sigma \subseteq N$ is polyhedral, then $\sigma^{\vee\vee} = \sigma$.*

*Proof.* For any set $S \subseteq N$ it is trivial to show $S^{\vee\vee\vee} = S^{\vee}$. The claim follows since $\sigma = S^{\vee}$ for a finite set $S \subseteq N$. □

**Proposition 14** (Fourier-Motkin elimination)**.** *If $\sigma \subseteq N$ is a polyhedral cone and $w \in N$, then $\sigma + \mathrm{Cone}(\{w\})$ is polyhedral too.*

*Proof.* Write $\sigma = S^{\vee}$ for a finite $S \subseteq M$. Define

$$T \;:=\; \{x \in S \mid 0 \leq \langle x, w\rangle\} \;\cup\; \{y\langle x, w\rangle - x\langle y, w\rangle \mid x, y \in S, 0 \leq \langle x, w\rangle, \langle y, w\rangle < 0\} \;\subseteq\; M.$$

Note that $T$ is finite. We claim $S^{\vee} + \mathrm{Cone}(\{w\}) = T^{\vee}$. Clearly $T \subseteq \mathrm{Cone}(S)$, hence $S^{\vee} = \mathrm{Cone}(S)^{\vee} \subseteq T^{\vee}$. Also clear is $w \in T^{\vee}$, hence $S^{\vee} + \mathrm{Cone}(\{w\}) \subset T^{\vee}$. For the reverse inclusion, let $v \in T^{\vee}$. We need to show $v - uw \in S^{\vee}$ for some $u \in \Bbbk_{\geq 0}$. If there is no $y \in S$ with both $\langle y, w\rangle < 0$ and $\langle y, v\rangle < 0$, we can easily see that $u = 0$ works, i.e. $T^{\vee} \subseteq S^{\vee}$. Otherwise, we set

$$u \;:=\; \max_{\substack{y \in S \\ \langle y, w\rangle < 0}} \frac{\langle y, v\rangle}{\langle y, w\rangle},$$

which is then well-defined and non-negative. We need to show $u\langle z, w\rangle \leq \langle z, v\rangle$ for all $z \in S$. If $\langle z, w\rangle < 0$, this follows from maximality of $u$. If $\langle z, w\rangle = 0$, the claim follows from $z \in T$ and $v \in T^{\vee}$. Lastly, assume $\langle z, w\rangle > 0$ and fix $y \in S$ such that $u = \frac{\langle y, v\rangle}{\langle y, w\rangle}$. In particular, $\langle y, w\rangle < 0$. Then the claim $u\langle z, w\rangle \leq \langle z, v\rangle$ is equivalent to

$$\langle y, v\rangle\langle z, w\rangle \geq \langle z, v\rangle\langle y, w\rangle.$$

But $y\langle z, w\rangle - z\langle y, w\rangle \in T$, hence the claim follows from $v \in T^{\vee}$. □

**Proposition 15.** *If $\sigma \in N$ is finitely generated, then it is polyhedral.*

*Proof.* Write $\sigma = \mathrm{Cone}(S)$ and use induction on the size of $S$: For $S = \emptyset$, we use 8 and the induction step is 14. □

**Proposition 16.** *For a finite set $S \subseteq N$, we have $S^{\vee\vee} = \mathrm{Cone}(S)$.*

*Proof.* The cone $\sigma := \mathrm{Cone}(S)$ is finitely generated, hence polyhedral by 15. Furthermore, we have $\sigma^{\vee\vee} = S^{\vee\vee}$. Now apply 13. □

**Proposition 17** (Polyhedral = Finitely generated)**.** *A pointed cone $\sigma \subseteq N$ is finitely generated iff it is polyhedral.*

*Proof.* The implication "⇒" is Proposition 15. For the converse, assume $\sigma = S^{\vee}$ for some finite $S \subseteq M$. Then $\mathrm{Cone}(S)$ is finitely generated, hence polyhedral by 15. Thus we find a finite $T \subseteq N$ with $\mathrm{Cone}(S) = T^{\vee}$. We obtain

$$\sigma = S^{\vee} = \mathrm{Cone}(S)^{\vee} = T^{\vee\vee} = \mathrm{Cone}(T),$$

where the last equality is due to 16. Thus $\sigma$ is finitely generated. □

**Proposition 18.** *If $\sigma \subseteq N$ is polyhedral, then $\sigma^{\vee} \subseteq M$ is polyhedral too.*

*Proof.* 12 + 17. □

**Proposition 19.** *Every polyhedral cone can be written as the minkowski sum of a pointy cone and a linear subspace.*

### 0.1.2 Operations

**Proposition 20.** *If $\sigma \subseteq N$ is polyhedral and $f : N -> M$ is a linear map, then $f.map\sigma \subseteq M$ is polyhedral too.*

**Proposition 21.** *If $\sigma, \tau \subseteq N$ are polyhedral, then $\sigma + \tau$ is polyhedral too.*

**Proposition 22.** *If $\sigma, \tau \subseteq N$ are polyhedral, the direct sum $\sigma \oplus_p \tau$ is polyhedral too.*

**Proposition 23.** *If $\sigma, \tau \subseteq N$ are polyhedral, their join is polyhedral too.*

**Proposition 24.** *If $\sigma, \tau \subseteq N$ are polyhedral, then $\sigma \cap \tau$ is polyhedral too.*

**Proposition 25.** *If $\sigma, \tau \subseteq N$ are polyhedral, then the cartesian product $\sigma \prod \tau$ is polyhedral too.*

**Definition 26** (Convex hull)**.** For a set $S \subseteq N$, the **convex hull of** $S$ is

$$\mathrm{Conv}(S) := \left\{ \sum_{u \in S} \lambda_u | \lambda_u \geq 0, \sum_u \lambda_u = 1 \right\}$$

**Proposition 27.** *If $\sigma, \tau \subseteq N$ are polyhedral, then the convex hull of the two is polyhedral too.*

**Proposition 28.** *Polyhedral cones for a lattice under intersections and convex hulls (its the same lattice as on pointed cones).*

### 0.1.3 Special cones

**Proposition 29.** *A simplicial cone $\sigma$ is a polyhedral cone that can be generated by $dim(\sigma)$ many generators.*

### 0.1.4 Faces and the Face Lattice

We define faces of polyhedral cones. The main goal is the partial order relation on the set of faces and showing that there is a bijection between faces of $\sigma$ and faces of $\sigma^\vee$.

**Definition 30.** For any subset $A \subseteq M$, we define the *perp space*

$$A^\perp \ := \ \{v \in N \ \mid \ \forall u \in A, \langle v, u \rangle = 0\},$$

which is a pointed cone in $N$. We also write $u^\perp := \{u\}^\perp$.

**Lemma 31.** *If $A \subseteq M$ is closed under negation (for instance, $A \subseteq M$ a $\Bbbk$-submodule), then $A^\perp = A^\vee$.*

*Proof.* $A^\perp \subseteq A^\vee$ is clear. Let $x \in A^\vee$ and $a \in A$. then $\langle x, a \rangle \geq 0$ and $-\langle x, a \rangle = \langle x, -a \rangle \geq 0$. Hence $\langle x, a \rangle = 0$ and $x \in A^\perp$. □

**Definition 32** (Face of a cone)**.** A *face* of a pointed cone $\sigma \subseteq N$ is the intersection of $\sigma$ with some hyperplane $u^\perp$ for $u \in \sigma^\vee$. We write $\tau \preceq \sigma$ if $\tau$ is a face of $\sigma$ and $\tau \prec \sigma$ if $\tau$ is a *proper* face, i.e. $\tau \preceq \sigma$ and $\tau \neq \sigma$.

**Proposition 33.** *For any subset $S \subseteq N$ and $u \in S^\vee$, we have*

$$\mathrm{Cone}(S) \cap u^\perp = \mathrm{Cone}(S \cap u^\perp).$$

*Proof.* Both inclusions should be a simple Span-induction. □

**Lemma 34.** *A face of a polyhedral cone is polyhedral.*

*Proof.* Use 33 and the fact that if $S$ is finite, so is $S \cap u^\perp$. □

**Lemma 35.** *A polyhedral cone has only finitely many faces.*

*Proof.* By 34, a face of a polyhedral cone $\sigma = \mathrm{Cone}(S)$ is of the form $\mathrm{Cone}(S')$ for some $S' \subseteq S$. Being finite, $S$ has only finitely many subsets, which shows the claim. □

**Lemma 36** (Intersection of faces)**.** *If $\sigma$ is a pointed cone, then the intersection of two faces of $\sigma$ is a again face of $\sigma$.*

*Proof.* Let $\tau = \sigma \cap u^\perp$ and $\tau' = \sigma \cap \tau'^\perp$ be faces. We claim that $\sigma \cap u^\perp \cap u'^\perp = \sigma \cap (u + u')^\perp$. The inclusion "$\subseteq$" is obvious. For the converse, suppose $v \in \sigma$ with $\langle v, u\rangle + \langle v, u'\rangle = 0$. Since $u, u' \in \sigma^\vee$, both summands are nonnegative, hence $\langle v, u\rangle = \langle v, u'\rangle = 0$. □

**Lemma 37.** *Let $\tau \preceq \sigma$ be a face of a pointed cone. Then $\tau^\vee = \tau^\perp + \sigma^\vee$.*

*Proof.* It's easy to show that $\tau = \mathrm{Span}_{\Bbbk}(\tau) \cap \sigma^\vee$. Dualising gives $\tau^\vee = \mathrm{Span}_{\Bbbk}(\tau)^\vee + \sigma^\vee$. Apply 31 to get $\mathrm{Span}_{\Bbbk}(\tau)^\vee = \mathrm{Span}_{\Bbbk}(\tau)^\perp = \tau^\perp$. □

**Lemma 38** (Face of a face)**.** *A face of a face of a polyhedral cone $\sigma$ is again a face of $\sigma$.*

*Proof.* Let $\tau = \sigma \cap u^\perp$ with $u \in \sigma^\vee$ and $\rho = \tau \cap v^\perp$ with $v \in \tau^\vee$. By 37, we can write $v = v_1 + v_2$ with $v_1 \in \tau^\perp$ and $v_2 \in \sigma^\vee$. Then $u + v_2 \in \sigma^\vee$ and we claim $\rho = \sigma \cap (u + v_2)^\perp$. For "$\subseteq$", let $x \in \rho = \tau \cap v^\perp$. Since $x \in \tau$, we get $\langle x, u\rangle = 0$ and $\langle x, v_1\rangle = 0$. Since $x \in v^\perp$, also $\langle x, v\rangle = 0$, which by $v = v_1 + v_2$ implies $\langle x, v_2\rangle = 0$. In total, $\langle x, u + v_2\rangle = 0$. For the converse, assume $x \in \sigma \cap (u + v_2)^\perp$, i.e. $\langle x, u\rangle + \langle x, v_2\rangle = 0$. Since both $u, v_2 \in \sigma^\vee$ and $x \in \sigma$, both summands are nonnegative, hence $\langle x, u\rangle = \langle x, v_2\rangle = 0$. This implies $x \in \tau$, thus also $\langle x, v_1\rangle = 0$. In total, $\langle x, v\rangle = \langle x, v_1\rangle + \langle x, v_2\rangle = 0$, thus $x \in \rho$. □

In particular, Lemma 38 establishes a partial order on the set of faces of a polyhedral cone $\sigma$ (written faces($\sigma$)). It is finite by 35.

**Lemma 39** (Face order is a lattice)**.** *Given a polyhedral cone, the partial order on its faces is a lattice. We call it the* face lattice *of the cone.*

*Proof.* Top is the entire cone, bot is the empty cone. □

**Lemma 40** (Face lattice is graded)**.** *Given a polyhedral cone, the partial order on its faces is graded.*

*Proof.* The grading is given by the number of faces of this face. □

**Definition 41** (Combinatorial Equivalence)**.** Two polyhedral cones are *combinatorially equivalent* if there is an order isomorphism between their face lattices.

**Proposition 42.** *Face lattices of dual cones are dual to each other.*

**Proposition 43.** *The face lattice of a face is isomorphic to a lower interval of the face lattice of a polyhedral cone.*

**Proposition 44.** *Linear transformaions preserve combinatorial equivalence. There is an iso that should be provided*

### 0.1.5 Polarity

**Definition 45** (Polar Cone)**.** For a pointed cone $\sigma$, the **polar cone of** $\sigma$ is the pointed cone

$$S^{\circ} := \{x \in \sigma \mid \forall y \in \sigma, \langle x, y\rangle \leq 0\}$$

## 0.2 Polytopes

**Definition 46** (Polytope)**.**
A **polytope** is a set that can be written as $\mathrm{Conv}(S)$ for some finite set $S$.